Arithmetic Applications of Hankel Determinants

Daniel James Sutherland

BMath(Hons)/BSc

Dissertation Submitted to
The School of Mathematical and Physical Sciences
at The University of Newcastle
for the Degree of
Doctor of Philosophy

Submitted: December 2014 Revised: February 2015

Statement of Originality

The thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to the final version of my thesis being made available worldwide when deposited in the University's Digital Repository, subject to the provisions of the Copyright Act 1968.

Candidate name: ____________________ Candidate signature: ____________________

Declaration by the candidate

I declare that the details provided above are correct.

Candidate Signature: ____________________

Endorsement by the supervisor

I, as the supervisor of the candidate, certify that the details provided above are correct.

Supervisor name: ____________________

Supervisor signature: ____________________ Date: ____________________

Acknowledgements

I would like to thank my supervisors Michael Coons, Wadim Zudilin and Jonathan Borwein for their guidance at various stages throughout my research. Their support has been invaluable and their enthusiasm has always given me confidence.

I thank Timothy Trudgian and Jean-Paul Allouche for examining my thesis and providing me with helpful feedback.

I thank my parents, Brian and Heather, who have always been eager to hear about my mathematical successes and setbacks, and my brother Josh for helping me to improve my explanatory skills.

I also thank my wonderful fiancée Skye, who has helped to bear the burdens of my research and who has always encouraged me to press on. I am grateful for her belief in my ability and her willingness to become familar with my work.

Lastly, I would also like to acknowledge the One from whom all things derive their order and beauty, my Lord of whom it is written:

I wisdom dwell with prudence, and find out knowledge of witty inventions.

— Proverbs 8:12 (KJV)

Contents

Abstract

This thesis focuses on the application of matrix determinants as a means of producing number-theoretic results. Motivated by an investigation of properties of the Riemann zeta function, we examine the growth rate of certain determinants of zeta values. We begin with a generalisation of determinants based on the Hurwitz zeta function, where we describe the arithmetic properties of its denominator and establish an asymptotic bound. We later employ a determinant identity to bound the growth of positive Hankel determinants. Noting the positivity of determinants of Dirichlet series allows us to prove specific bounds on determinants of zeta values in particular, and of Dirichlet series in general. Our results are shown to be the best that can be obtained from our method of bounding, and we conjecture a slight improvement could be obtained from an adjustment to our specific approach.

Within the course of this investigation we also consider possible geometric properties which are necessary for the positivity of Hankel determinants, and we examine the role of Hankel determinants in irrationality proofs via their connection with Padé approximation.

Notation

$\mathbb{N}$	The set of natural numbers $\{1, 2, 3, \ldots\}$.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^2$	The set of ordered pairs (x, y) of real numbers $x, y \in \mathbb{R}$.
$\mathfrak{S}_n$	The symmetric group of order n.
$\mathbb{Z}$	The set of integers.
$\mathrm{adj}(M)$	The adjugate of the square matrix M.
$\mathrm{denom}(P(x))$	The integer denominator of the coefficients of $P(x) \in \mathbb{Q}[x]$.
$\mathrm{lcm}\{n_1, n_2, \ldots\}$	The least common multiple of a set of natural numbers.
$\phi(n)$	The number of positive integers less than n that are coprime to n.
$\Re(s)$	The real part σ of the complex number $s = \sigma + it$, with $\sigma, t \in \mathbb{R}$.
$\mathrm{sgn}(\pi)$	The sign of a permutation $\pi \in \mathfrak{S}_n$.
$\mathrm{sgn}(x)$	The sign of $x \in \mathbb{R}$, equal to $x/\lvert x\rvert$, $x \neq 0$.
$f(n) = \mathcal{O}(g(n))$	The ratio $\lvert f(n)/g(n)\rvert$ is bounded as n tends to ∞, $g(n) \neq 0$.
$f(n) = o(g(n))$	The ratio $\lvert f(n)/g(n)\rvert$ tends to 0 as n tends to ∞, $g(n) \neq 0$.
$f(n) \asymp g(n)$	Equivalent to $f(n) = \mathcal{O}(g(n))$ and $g(n) = \mathcal{O}(f(n))$.
C, c	A constant (different subscripts denote different constants).
p	A prime $p \in \mathbb{N}$.